MMP Learning Seminar, Week 87:
Boundedness of exceptional Fano varieties.

Theorem (Hacon-Xu): Fix $I \subseteq [0,1] \cap \mathbb{Q}$ satisfying the DCC. Let d be a positive integer. Assume that:

- (X,B) is a d-dim klt pair,
- the coeff of B are in I,
- the divisor B is big, and
- $K_X + B \sim_{\mathbb{Q}} 0$.

Global ACC implies that the index of K_X+B is controlled in terms of I & d.

the log discr are bounded away from 0.

Then, X belongs to a bounded family.

Theorem 1.4: Let d be a natural number. Let $\varepsilon, \delta > 0$. Consider the set of pairs (X,B) such that:

- (X,B) is ε-lc & d-dimensional,
- B is big & $K_X+B \sim_{\mathbb{R}} 0$, and
- coeff $B \geq \delta$.

Then such set of X's forms a bounded family.

Remark: A Fano type variety belongs to a bounded family if it admits a bounded klt complement.

def: X is exceptional Fano, for every $0 \leq B \sim_{\mathbb{Q}} -K_X$ the pair (X, B) is klt. $\leftrightarrow$ is ε-klt (for some ε only dep on d)

τ-lc: $|-mK_X|$ defines a birational map for some ct that only depends on τ & d. $0 \leq \Gamma \in |-mK_X|$, $(X, \Gamma/m)$ is a klt m-complement. $m(K_X + \Gamma/m) \sim 0$

Apply HX to conclude that X belongs to a bounded family

not τ-lc: $Y \xrightarrow{\phi} X$, that extracts a prime divisor $D \subseteq Y$.

$\phi^* K_X = K_Y + eD$, $e > 1-\tau$.

$-(K_Y + eD)$-MMP. terminate with $Y' \longrightarrow Z'$.

and $K_{Y'} + \tilde{e}D'$ is trivial for this morphism. $\tilde{e} \geq e$.

Apply 1.4 to show that $\tilde{e}$ belongs to a finite family.

(*) If $\dim Z' > 0$, then we can pull-back a comp from Z'.

If $\dim Z' = 0$, then we can apply HX to conclude that Y', hence Y & X belong to a bounded family

Needs complements in $\dim \leq d-1$.

$\Downarrow$

Needs relative complements in $\dim \leq d$.

This is what we will cover next week.

Theorem 1.11: Let d & p be natural numbers. $\mathcal{R}\subseteq[0,1]$ be finite. Let $(X,B+M)$ be a generalized Fano type pair. The divisor pM' is nef & Cartier. The coeff of B are in $\Phi(\mathcal{R}) = \{1-\frac{r}{m} \mid r\in\mathcal{R},\ m\in\mathbb{Z}_{>0}\}\cup\{1\}$. And $\underline{(X,B+M)}$ is exceptional. Then the pairs (X,B) belong to a bounded family.

Lemma 1: There $\varepsilon>0$ only depending on the previous data such that if $0\leq P\sim_{\mathbb{Q}} -(K_X+B+M)$, then $(X,B+P+M)$ is ε-glc.

Proof: $(X_i,B_i+P_i+M_i)$ is ε_i-glc with $\varepsilon_i\to 0$.

$X_i' \xrightarrow{\varphi_i} X_i$ extracting D_i'.

$$\varphi_i^*(K_{X_i}+B_i+P_i+M_i) = K_{X_i'}+B_i'+\boxed{P_i'}+\overset{1.}{(1-\varepsilon_i)}D_i'+M_i.$$

Find a new model $X_i' \dashrightarrow X_i''$

here

$(X_i'',B_i''+D_i''+M_i'')$ is glc

$-(K_{X_i''}+B_i''+\underline{D_i''}+M_i'')$ is semiample.

bounded comp $\Longrightarrow$ bounded non-klt comp for (X_i,B_i+M_i) □

these two will need to be perturbed

Remark: Even if $(X,B+M)$ is exceptional Fano type it could happen that X is not exceptional.

Ex: $(\mathbb{P}^1,\{0\}+\{\infty\})$, $(\mathbb{P}^1,\frac{2}{3}\{0\}+\frac{3}{4}\{1\}+\frac{4}{5}\{\infty\})$

↓ log Fano exceptional.

Lemma 2: Let X be a Fano type variety & $\mathbb{Q}$-factorial. $-(K_X+B)$ is nef. $D\neq 0$ effective on X. Then, there exists a $(-D)$-MMP ending with a non-bir contraction $X' \longrightarrow T'$ s.t. $-(K_{X'}+B'+tD')$ is globally nef & numerically trivial over T' for some $t\geq 0$. Furthermore, the intersection of K_X+B+tD with each extremal ray of this MMP is non-neg.

Sketch: $s\geq 0$ the largest number for which $-(K_X+B+sD)$ is nef.

$(K_X+B+sD)\cdot R>0$, $D\cdot R>0$, R is extremal.

Proceeding inductively, we are getting a $(-D)$-MMP. $\square$.

Lemma 3: Assume boundedness of complements in dim $d-1$ & boundedness of relative complements in dim d. Then boundedness of exceptional Fanos hold in dim d.

Proof: X_i s.t $a_i =$ min log discr of $X_i \longrightarrow 0$.

$X_i' \xrightarrow{\varphi_i} X_i$, $\varphi_i^* K_{X_i} = K_{X_i'} + e_i D_i'$, e_i close to 1. $D_i' \neq 0$

Apply Lemma 2 to D_i'.

$$\begin{tikzcd} X_i' \arrow[r, dashed] & X_i'' \arrow[r, "\psi_i"] & T_i \end{tikzcd}$$ non-bir contraction

- $(K_{X_i''} + e_i D_i'' + t_i D_i'')$ is globally nef

  & numerically trivial over T_i for some $t_i \geq 0$

Applying 1.4 to the general fibers of ψ_i we conclude that $e_i + t_i$ belong to a finite set.

If $\dim T_i > 0$, we can lift a complement from T_i by using the cbf.

If $\dim T_i = 0$, we conclude that X_i'' belongs to a bounded family by using HX.

By pulling back to X_i we obtain a bounded klt complement on X_i. $\square$

X Fano type, $(X, B+M)$ gen klt exceptional.

coeff $B \in \mathfrak{D}(\mathcal{R})$

pM is b-nef & Cartier.

X d-dimensional

Lemma 4 (Bound on exc thresholds):

$(X, B+\beta M)$ is exceptional for β close enough to 1.

Proof: $(X_i, B_i + \beta_i M_i)$ not exc $\quad \beta_i \longrightarrow 1.$

$(X_i, B_i + \beta_i M_i + P_i)$ not gen klt. $\quad P_i$ is nef.

$t_i = \operatorname{lct}(X_i, B_i + \beta_i M_i ; P_i) \leq 1. \qquad \Omega_i = B_i + t_i P_i$

$(X_i', \Omega_i' + \beta_i M_i') \longrightarrow (X_i, \Omega_i + \beta_i M_i). \qquad \boxed{\lfloor \Omega_i' \rfloor \neq \emptyset}$

We can increase $\beta_i \longrightarrow 1.$ and keep the glc condition and anti-nefness in a diff birational model.

$$\begin{tikzcd} X_i' \arrow[r, dashed] \arrow[d] & X_i'' \arrow[l, bend right] \\ X_i \end{tikzcd}$$

$(X_i'', \Omega_i'' + M_i'')$ is glc

&

$-(K_{X_i} + \Omega_i'' + M_i'')$ is nef.

$\downarrow$ this admits a comp.

This induces a non gklt comp for $(X_i, B_i + M_i)$ $\quad \rightarrow \leftarrow$

$\square$

Lemma 5 (boundedness of volume):

$(X, B+M)$ gen klt, $K_X + B + M \sim_{\mathbb{R}} 0$. Then $\mathrm{vol}(-K_X) \leq v$.

Proof: $\mathrm{vol}(-K_{X_i}) \longrightarrow \infty$

$\quad \| $

$\quad v_i$

X_i is $\mathbb{Q}$-factorial.

$K_{X_i} + 3dp M_i$ is big $\leftarrow$ follows from cone Theorem

$$\mathrm{vol}(-K_{X_i}) < \mathrm{vol}(-K_{X_i} + \underbrace{K_{X_i} + 3dpM_i}_{\text{big}}) = \mathrm{vol}(3dpM_i)$$

$\mathrm{vol}(\delta_i M_i) > (2d)^d$ for $\delta_i \longrightarrow 0$. $\quad \alpha_i = 1 - \delta_i$

$$\mathrm{vol}(-(K_{X_i} + B_i + \alpha_i M_i)) > (2d)^d.$$

$\downarrow$

not exceptional $\quad \longrightarrow\!\!\leftarrow$

$(X_i, B_i + M_i)$ is exceptional

$\square$

Lemma 6 (Bound on log canonical thresholds):

Fix $d, p, \ell \in \mathbb{N}$ and $\Phi \subseteq [0,1]$. $(X, B+M)$ is exceptional gen pair & X is Fano type. Then for any $L \in |-\ell K_X|$
Then (X, tL) is klt for some $t \geq t(d, p, \ell, \Phi)$.

Rem: X may not be exceptional !!

Proof: (X_i, B_i+M_i) as in the statement.

$(X_i, t_i L_i)$ not klt, $t_i \longrightarrow 0$.

$$(*) \quad K_{X_i} + 3dp M_i \sim_{\mathbb{Q}} \boxed{-\frac{1}{\ell} L_i + 3dp M_i} \text{ is big}$$

$(X_i, B_i + \beta_i M_i)$ is exceptional.

$\boxed{s_i} = \operatorname{glct}((X_i, B_i + \beta_i M_i); L_i)$. Assume $\boxed{s_i < \frac{1-\beta}{3dp\ell}}$ $(**)$

Study $(X_i, B_i + s_i L_i + \beta_i M_i)$.

$$-(K_{X_i} + B_i + s_i L_i + \beta_i M_i) = \underbrace{-(K_{X_i} + B_i + M_i)}_{\text{pseff}} + \underbrace{(1-\beta) M_i - s_i L_i}_{\text{big}}$$

big.

$0 \leq P_i \sim_{\mathbb{R}} -(K_{X_i} + B_i + sL_i + \beta_i M_i)$

$(X_i, B_i + s_i L_i + \beta_i M_i + P_i)$ is not gklt

this contradicts the exceptionality of $(X_i, B_i + \beta_i M_i)$ $\square$

Summary.

X Fano type. $(X, B+M)$ exceptional.

- X is d-dim
- coeff $(B) \in \Phi(R)$
- pM is b-nef & Cartier.

- $(X, B+\beta M)$ is exceptional for β close to 1.
- $\mathrm{Vol}(-K_X) \leq v$.
- The log canonical thresholds of elements in $|-\ell K_X|$ are controlled away from zero for ℓ fixed.

Proposition: Assume existence of rel comp in dim $\leq d$ & existence of proj complements in dim $\leq d-1$

Fix $d, m, v \in \mathbb{N}$ & t_ℓ a sequence of reals.

Let $\mathcal{P}$ be the set of varieties X s.t:

produce a new bounded klt comp

- X is weak Fano of dim d,
- K_X has a m-complement,
- $|-mK_X|$ defines a bir map,
- $\mathrm{vol}(-K_X) \leq v$

produce a log bounded famly

- for any $\ell \in \mathbb{N}$, and any $L \in |-\ell K_X|$, the pair $(X, t_\ell L)$ is klt

Then $\mathcal{P}$ forms a bounded family

Sketch: Step 1: We use 3 + 4 to show that there exists a log bir boundiy family

$(\overline{W}, \Sigma_{\overline{W}}) \dashrightarrow X.$

$M \in |-mK_X|$ general, supp $\Sigma_{\overline{W}}$ contain supp of M

$M_{\overline{W}} = A_{\overline{W}} + R_{\overline{W}}.$

Step 2: We may assume $B^+ = \frac{1}{m} A + \frac{1}{m} R.$

Step 3: $\ell A_{\overline{W}} \sim G_{\overline{W}} \geq 0$ contaiy $\Sigma_{\overline{W}}$ in its supp.

$\ell A \sim G \geq 0.$ $G + \ell R \in |-\ell m K_X|.$

$(X, t(G + \ell R))$ where $t = t_{\ell m}.$

- $(X, \frac{1}{\ell m}(G+\ell R))$ is lc , $\Omega = \frac{1}{\ell m}(G+\ell R)$.
- $(X, t(G+\ell R))$ is not lc, strongly non-exc,

there exists a n-complement..

$\boxed{\Omega} \geq t(G+\ell R)$, (X,Ω) is lc $n(K_X+\Omega)\sim 0$.

n-comp Ω.

<u>Step 4 & 5</u>: $\Delta = B^f + \frac{t}{m}A - \boxed{\frac{t}{\ell m}G}$ sub-ε-lc complement.

$\Theta = \frac{1}{2}\Delta + \frac{1}{2}\Omega.$ ← ~~sub~~-klt complement

We obtained a bounded klt comp for X □.

Theorem: Assume existence of rel comp in dim $\leq d$ & existence of proj complements in dim $\leq d-1$

Then d-dim exc gen Fano type pairs $(X, B+M)$ satisfy that (X,B) is log bounded

Sketch: Step 1: We may assume coeff B are finite + it suffices to prove that X is bounded

Step 2: It suffices to produce a bounded klt comp for X.

Step 3: Create a complement for X,

X ε-lc, $|-mK_X|$ defines a birational map.

$\phi^*(-mK_X) \sim A + R$.

Step 4: $\Delta = \frac{1}{2}B + \frac{1}{2m}R$, & $N = \frac{1}{2}M + \frac{1}{2m}A$

Case 1: $(X, \Delta+N)$ is exceptional

$|-r(K_X+\Delta+N)|$ is birational

$\longrightarrow$ reduce to the case in which $K_X+B+M \sim_{\mathbb{R}} 0$, M big.

Case 2: $(X, \Delta+N)$ is not exc

we can produce a bounded comp

Step 5: $(X, B+M)$ gen klt, $K_X + B + M \sim_{\mathbb{R}} 0$.

M is big

- K_X has a m-comp.
- $|-mK_X|$ defines a bir map.
- $\mathrm{vol}(-K_X) \leq v$.
- there exist t_ℓ s.t for any $L \in |-\ell K_X|$ $(X, t_\ell L)$ is klt

X belongs to a bounded family.

$\square$.

Section: Comp in dim $d-1$ $\Longrightarrow$ Relative comp in dim d.